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Frames

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Frames

1. Frames and Riesz Bases
2. Translation-Invariant Dyadic Wavelet Transform
3. Subsampled Wavelet Frames

Introduction

- A signal representation may provide "analysis" coefficients that are inner products with a family of vectors, or "synthesis" coefficients that compute an approximation by recombining a family of vectors.

- Frames are families of vectors where "analysis" and "synthesis" representations are stable. Signal reconstructions are computed with a dual frame.

- Frames are potentially redundant and thus more general than bases, with a redundancy measured by frame bounds. They provide the flexibility needed to build signal representations with unstructured families of vectors.

- Complete and stable wavelet Fourier transforms are constructed with frames of wavelets.

Frames

1 **Frames and Riesz Bases**

Stable Analysis and Synthesis Operators

Frame analysis operator

- The frame theory was originally developed to reconstruct band-limited signals from irregularly spaced samples. The following frames definition gives an energy equivalence to invert the operator Φ defined by

$$\forall n \epsilon \Gamma, \Phi f[n] = \langle f, \phi_n \rangle.$$

If satisfied

- **Definition 5.1:** *Frame and Riesz Basis.* The sequence $\{\phi_n\}_{n\epsilon\Gamma}$ is a frame of $\mathbf{H}$ if there exist two constants $B \geq A > 0$ such that

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \leq \sum_{n\epsilon\Gamma} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2.$$

When $A = B$, the frame is said to be tight. If the $\{\phi_n\}_{n\epsilon\Gamma}$ are linearly independent then the frame is not redundant and is called a *Riesz basis*.

Then Φ is called a *frame analysis operator*.

Stable Analysis and Synthesis Operators

Frame analysis operator

Definition 5.1:

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \leq \sum_{n\epsilon\Gamma} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2.$$

- It is a necessary and sufficient condition guaranteeing that Φ is invertible on its image space, with a bounded inverse.

- Thus, a frame defines a complete and stable signal representation, which may also be redundant.

Stable Analysis and Synthesis Operators

Frame synthesis operator

- Let us consider the space of finite energy coefficients

$$\ell^2(\Gamma) = \{a : \|a\|^2 = \sum_{n\epsilon\Gamma} |a[n]|^2 < +\infty\}.$$

- The adjoint $\Phi^*$of Φ is defined over $\ell^2(\Gamma)$ and satisfies for any $f \epsilon \mathbf{H}$ and $a \in \ell^2(\Gamma)$:

$$\langle \Phi^* a, f \rangle = \langle a, \Phi f \rangle = \sum_{n\epsilon\Gamma} a[n] \langle f, \phi_n \rangle^* = \left\langle \sum_{n\epsilon\Gamma} a[n]\, \phi_n, f \right\rangle.$$

- The ***frame synthesis operator***:

$$\Phi^* a = \sum_{n\epsilon\Gamma} a[n]\, \phi_n.$$

Let $a = \Phi f$

$$\Phi^* \Phi f = \sum_{n\epsilon\Gamma} \Phi f[n]\, \phi_n = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle\, \phi_n.$$

$$\langle \Phi^* \Phi f, f \rangle = \langle \Phi f, \Phi f \rangle = \|\Phi f\|^2 = \sum_{n\epsilon\Gamma} |\langle f, \phi_n \rangle|^2$$

Stable Analysis and Synthesis Operators

Frame synthesis operator

Definition 5.1(Frame Condition):

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \le \sum_{n\epsilon\Gamma} |\langle f, \phi_n\rangle|^2 \le B\|f\|^2.$$

$$\langle \Phi^*\Phi f, f\rangle = \langle \Phi f, \Phi f\rangle = \|\Phi f\|^2 = \sum_{n\epsilon\Gamma} |\langle f, \phi_n\rangle|^2$$

Definition 5.1(Frame Condition):

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \le \langle \Phi^*\Phi f, f\rangle \le B\|f\|^2.$$

in finite $\Downarrow$ dimension

$$A\|f\|^2 \le \langle \lambda_i f, f\rangle \le B\|f\|^2$$

$$\Longrightarrow \quad A\|f\|^2 \le \lambda_i \|f\|^2 \le B\|f\|^2$$

$$\Longrightarrow \quad A \le \lambda_i \le B$$

$$\Phi^*\Phi f = \lambda_i f$$

λ_i is the eigenvalues

- A and B are the smallest and largest eigenvalues in finite dimension.
- The eigenvalues are also called *singular values* of Φ or *singular spectrum*.
- A and B are the infimum and supremum values of the spectrum of the symmetric operator $\Phi^*\Phi$.

Stable Analysis and Synthesis Operators

Frame synthesis operator

Definition 5.1(Frame Condition):

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \leq \sum_{n\epsilon\Gamma} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2.$$

frame synthesis operator

$$\Phi^* a = \sum_{n\epsilon\Gamma} a[n]\, \phi_n$$

Stable!

frame analysis operator

$$\Phi f[n] = \langle f, \phi_n \rangle$$

- **Theorem 5.1:** The family $\{\phi_n\}_{n\epsilon\Gamma}$ is a frame with bounds $B \geq A > 0$ if and only if

$$\forall a \epsilon \mathbf{Im}\Phi, A\|a\|^2 \leq \left\| \sum_{n\epsilon\Gamma} a[n]\, \phi_n \right\|^2 \leq B\|a\|^2.$$

Stable Analysis and Synthesis Operators

Frame synthesis operator

- Proof of **Theorem 5.1:**

Theorem 5.1:

$\forall a \epsilon \mathbf{Im}\Phi, A\|a\|^2 \leq \|\sum_{n\epsilon\Gamma} a[n]\, \phi_n\|^2 \leq B\|a\|^2.$

$$\|\sum_{n\epsilon\Gamma} a[n]\, \phi_n\|^2$$
$$= \|\Phi^* a\|^2 = \langle \Phi^* a, \Phi^* a\rangle$$
$$= \langle a, \Phi\Phi^* a\rangle = \langle \Phi\Phi^* a, a\rangle$$

$\Updownarrow$

Theorem 5.1:

$\forall a \epsilon \mathbf{Im}\Phi, \quad A\|a\|^2 \leq \langle \Phi\Phi^* a, a\rangle \leq B\|a\|^2.$

$\Leftrightarrow$ A and B are the infimum and supremum values of the spectrum of $\Phi\Phi^*$

$\Updownarrow$?

Definition 5.1:

$\forall f \epsilon \mathbf{H}, \quad A\|f\|^2 \leq \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2.$

$\Leftrightarrow$ A and B are the infimum and supremum values of the spectrum of $\Phi^*\Phi$

- In finite dimension, the maximum and minimum eigenvalues of $\Phi\Phi^*$and $\Phi^*\Phi$ on $\mathbf{Im}\Phi$ are identical.($\mathbf{Im}\Phi$ is the image space of all Φf)

Stable Analysis and Synthesis Operators

Frame synthesis operator

- When the frame vectors are normalized $\|\phi_n\| = 1$, the frame redundancy is measured by the frame bounds A and B.

> **Theorem 5.2:** In a space of finite dimension N, a frame of $P \geq N$ normalized vectors has frame bounds A and B, which satisfy
> $$A \leq \frac{P}{N} \leq B.$$
> For a tight frame $A = B = P/N$.

Proof: $\Phi\Phi^* a[p] = \langle \Phi^* a, \phi_p \rangle = \langle \sum_{n\epsilon\Gamma} a[n]\, \phi_n, \phi_p \rangle = \sum_{n\epsilon\Gamma} a[n] \langle \phi_n, \phi_p \rangle$

$$\Longrightarrow \Phi\Phi^* = \begin{bmatrix} \langle \phi_1, \phi_1 \rangle & \cdots & \langle \phi_P, \phi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_P \rangle & \cdots & \langle \phi_P, \phi_P \rangle \end{bmatrix} \Longrightarrow \mathrm{tr}(\Phi\Phi^*) = \sum_{n=1}^{p} |\langle \phi_n, \phi_n \rangle|^2 = P$$

Since $\mathrm{tr}(\Phi\Phi^*) = \mathrm{tr}(\Phi^*\Phi)$, and $AN \leq \mathrm{tr}(\Phi^*\Phi) = \sum_{i=1}^{N} \lambda_i \leq BN$

$$\Longrightarrow AN \leq P \leq BN \Longrightarrow A \leq \frac{P}{N} \leq B$$

$$\boxed{A \leq \lambda_i \leq B}$$

Stable Analysis and Synthesis Operators

Redundancy

Theorem 5.2: In a space of finite dimension N, a frame of $P \geq N$ normalized vectors has frame bounds A and B, which satisfy

$$A \leq \frac{P}{N} \leq B.$$

For a tight frame $A = B = P/N$.

- If $\{\phi_n\}_{n\epsilon\Gamma}$ is a normalized Riesz basis and is therefore linearly independent, then it proves that $A \leq 1 \leq B$. This result remains valid in infinite dimension.

- The frame is orthonormal if and only if $B = 1$, in which case $A = 1$.

- One can verify that a finite set of N vectors $\{\phi_n\}_{1 \leq n \leq N}$ is always a frame of the space $\mathbf{V}$ generated by linear combinations of these vectors.

- When N increases, the frame bounds A and B may go respectively to 0 and $+\infty$. This illustrates the fact that in infinite dimensional spaces, a family of vectors may be complete and not yield a stable signal representation.

Stable Analysis and Synthesis Operators

Redundancy

- **Example 5.1:** Let $\{g_1, g_2\}$ be an orthonormal basis of an $N = 2$ two-dimensional plane $\mathbf{H}$. The $P = 3$ normalized vectors

$$\phi_1 = g_1,\ \phi_2 = -\frac{g_1}{2} + \frac{\sqrt{3}}{2} g_2,\ \phi_3 = -\frac{g_1}{2} - \frac{\sqrt{3}}{2} g_2.$$

have equal angles of $2\pi/3$ between each other. For any $f \epsilon \mathbf{H}$,

$$\sum_{n=1}^{3} |\langle f, \phi_n \rangle|^2 = \frac{3}{2} \|f\|^2.$$

Thus, these three vectors define a tight frame with $A = B = 3/2$.

Stable Analysis and Synthesis Operators

Core Equations

$$\Phi f[n] = \langle f, \phi_n \rangle$$

$$\Phi^* a = \sum_{n \epsilon \Gamma} a[n]\, \phi_n$$

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \leq \sum_{n \epsilon \Gamma} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2.$$

$$\forall a \epsilon \mathbf{Im}\Phi, A\|a\|^2 \leq \left\| \sum_{n \epsilon \Gamma} a[n]\, \phi_n \right\|^2 \leq B\|a\|^2$$

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \leq \langle \Phi^* \Phi f, f \rangle \leq B\|f\|^2.$$

$$\forall a \epsilon \mathbf{Im}\Phi, A\|a\|^2 \leq \langle \Phi \Phi^* a, a \rangle \leq B\|a\|^2.$$

Dual Frame and Pseudo Inverse

Pseudo Inverse

- The reconstruction of f from its frame coefficients $\Phi f[n]$ is calculated with a pseudo inverse . This pseudo inverse is a bounded operator that implements a dual-frame reconstruction. For Riesz bases, this dual frame is a biorthogonal basis.

- **Theorem 5.3:** If $\{\phi_n\}_{n\epsilon\Gamma}$ is a frame but not a Riesz basis, then Φ admits an infinite number of left inverses.

- **Proof:**

ImU： the image space of all Uf and by
NullU： the null space of all h, such that $Uh = 0$.
$\mathbf{Null}\Phi^* = (\mathbf{Im}\Phi)^\perp$ is the orthogonal complement of $\mathbf{Im}\Phi$ in $\ell^2(\Gamma)$.

If Φ is a frame and not a Riesz basis, then $\{\phi_n\}_{n\epsilon\Gamma}$ is linearly dependent

$$\Longrightarrow \exists a \neq \mathbf{0}, \Phi^* a = \sum_{n\epsilon\Gamma} a[n]\, \phi_n = 0 \Longrightarrow \exists a \neq \mathbf{0}, a \in \mathbf{Null}\Phi^* = (\mathbf{Im}\Phi)^\perp$$

Dual Frame and Pseudo Inverse

Pseudo Inverse

- **Proof:**

$\because$ If $\Phi f = 0 \Longrightarrow A\|f\|^2 \leq \|\Phi f\|^2 = 0 \ (A > 0) \Longrightarrow f = 0$

$\therefore$ Φ admits a left inverse.

There is an infinite inverses since the restriction of a left inverse to $(\mathbf{Im}\Phi)^{\perp} \neq \{0\}$ may be any arbitrary linear operator.

- The more redundant the frame $\{\phi_n\}_{n\epsilon\Gamma}$, the larger the orthogonal complement $(\mathbf{Im}\Phi)^{\perp}$ of $\mathbf{Im}\Phi$ in $\ell^2(\Gamma)$. The pseudo inverse Φ^+, is defined as the left inverse that is zero on $(\mathbf{Im}\Phi)^{\perp}$:

$$\forall f\epsilon\mathbf{H},\ \Phi^+\Phi f = f \quad \text{and} \quad \forall a\epsilon(\mathbf{Im}\Phi)^{\perp},\ \Phi^+ a = 0.$$

- How to compute this pseudo inverse? $\longrightarrow$ **Theorem 5.4**

Dual Frame and Pseudo Inverse

Pseudo Inverse

- **Theorem 5.4:** Pseudo Inverse. If Φ is a frame operator, then $\Phi^*\Phi$ is invertible and the pseudo inverse satisfies

$$\Phi^+ = (\Phi^*\Phi)^{-1}\Phi^*$$

- **Proof:**

a) $\because$ If $\Phi^*\Phi f = 0 \Longrightarrow A\|f\|^2 \leq \sum_{n\epsilon\Gamma}|\langle f, \phi_n\rangle|^2 = \langle \Phi^*\Phi f, f\rangle = 0\ (A > 0) \Longrightarrow f = 0$

$\therefore$ $\Phi^*\Phi$ is invertible. For all $\forall f \epsilon \mathbf{H},\quad (\Phi^*\Phi)^{-1}\Phi^*\Phi f = f$

$\Longrightarrow\ (\Phi^*\Phi)^{-1}\Phi^*$ is a left inverse.

b) $(\mathbf{Im}\Phi)^{\perp} = \mathbf{Null}\Phi^* \Longrightarrow \forall a \epsilon (\mathbf{Im}\Phi)^{\perp}, \Phi^* a = 0$

$\Longrightarrow \forall a \epsilon (\mathbf{Im}\Phi)^{\perp}, (\Phi^*\Phi)^{-1}\Phi^* a = 0$

Pseudo inverse definition:

$\forall f \epsilon \mathbf{H},\ \Phi^+\Phi f = f$ and

$\forall a \epsilon (\mathbf{Im}\Phi)^{\perp},\ \Phi^+ a = 0$.

From a) and b), $(\Phi^*\Phi)^{-1}\Phi^*$ is the pseudo inverse.

Dual Frame and Pseudo Inverse

Dual Frame

- The pseudo inverse of a frame operator implements a reconstruction with a dual frame $\longrightarrow$ **Theorem 5.5**

- **Theorem 5.5:** Let $\{\phi_n\}_{n\epsilon\Gamma}$ be a frame with bounds $B \geq A > 0$. The dual operator defined by

$$\forall n\epsilon\Gamma, \widetilde{\Phi} f[n] = \langle f, \tilde{\phi}_n \rangle \quad \text{with} \quad \tilde{\phi}_n = (\Phi^* \Phi)^{-1} \phi_n$$

satisfies

$$\widetilde{\Phi}^* = \Phi^+ \qquad \text{(a)}$$

and thus

$$f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n \qquad \text{(b)}$$

It defines a dual frame as

$$\forall f \epsilon \mathbf{H}, \frac{1}{B} \|f\|^2 \leq \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle^2 \leq \frac{1}{A} \|f\|^2 \qquad \text{(c)}$$

If the frame is tight (i.e. $A = B$), then $\tilde{\phi}_n = A^{-1} \phi_n$.

Dual Frame and Pseudo Inverse

Dual Frame

- **Proof: (a)(b)**

$$\tilde{\Phi}f[n] = \langle f, \tilde{\phi}_n \rangle$$
$$= \langle f, (\Phi^*\Phi)^{-1}\phi_n \rangle$$
$$= \langle (\Phi^*\Phi)^{-1} f, \phi_n \rangle$$
$$= \Phi(\Phi^*\Phi)^{-1} f[n]$$
$$\Longrightarrow \tilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}$$
$$\Longrightarrow \tilde{\Phi}^* = (\Phi^*\Phi)^{-1}\Phi$$
$$\because \ \Phi^+ = (\Phi^*\Phi)^{-1}\Phi^*$$
$$\therefore \ \tilde{\Phi}^* = \Phi^+ \longrightarrow \text{(a) proved}$$

$$\tilde{\Phi}^*\Phi = \Phi^+\Phi = \mathrm{Id} \Longrightarrow \Phi^*\tilde{\Phi} = \mathrm{Id}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$f = \tilde{\Phi}^*\Phi f \qquad\qquad f = \Phi^*\tilde{\Phi} f$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

$\longrightarrow$(b) proved

Dual Frame and Pseudo Inverse

Dual Frame

- **Proof: (c)**

Definition 5.1:

$$\forall f \epsilon \mathbf{H}, A\|f\|^2 \le \langle \Phi^*\Phi f, f\rangle \le B\|f\|^2.$$

$$\Longrightarrow \forall f \epsilon \mathbf{H}, A \le \lambda_i \le B$$

$$\Downarrow$$

$$\forall f \epsilon \mathbf{H}, B^{-1}\|f\|^2 \le \langle (\Phi^*\Phi)^{-1} f, f\rangle \le A^{-1}\|f\|^2 \quad \Longleftarrow \forall f \epsilon \mathbf{H}, \frac{1}{B} \le \frac{1}{\lambda_i} \le \frac{1}{A}$$

λ_i is the eigenvalues of $\Phi^*\Phi$

$\frac{1}{\lambda_i}$ is the eigenvalues of $(\Phi^*\Phi)^{-1}$

$$\because \left\|\tilde{\Phi} f\right\|^2 = \langle \Phi(\Phi^*\Phi)^{-1} f, \Phi(\Phi^*\Phi)^{-1} f\rangle$$

$$= \langle f,\ (\Phi^*\Phi)^{-1}\Phi^*\Phi(\Phi^*\Phi)^{-1} f\rangle$$

$$= \langle f,\ (\Phi^*\Phi)^{-1} f\rangle$$

$$\therefore \forall f \epsilon \mathbf{H}, \frac{1}{B}\|f\|^2 \le \left\|\tilde{\Phi} f\right\|^2 = \sum_{n \epsilon \Gamma} \langle f, \tilde{\phi}_n\rangle^2 \le \frac{1}{A}\|f\|^2 \quad \longrightarrow \text{(c) proved}$$

Dual Frame and Pseudo Inverse

Dual Frame

Dual frame vector

Frame vector

$$f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

$$= \sum_{n\epsilon\Gamma} \Phi f[n] \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \tilde{\Phi} f[n] \phi_n$$

frame coefficient

Dual frame coefficient

- Theis theorem proves that f is reconstructed from frame coefficients $\Phi f[n] = \langle f, \phi_n \rangle$ with the dual frame $\{\tilde{\phi}_n\}_{n\epsilon\Gamma}$.
- The synthesis coefficients of f in $\{\phi_n\}_{n\epsilon\Gamma}$ are the dual-frame coefficients $\tilde{\Phi} f[n] = \langle f, \tilde{\phi}_n \rangle$.
- If the frame is tight (i.e. $\tilde{\phi}_n = A^{-1}\phi_n$) then both decompositions are identical:

$$f = \frac{1}{A} \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \phi_n$$

Dual Frame

Biorthogonal Bases

- A Riesz basis is a frame of vectors that are **linearly independent**, so its dual frame is also linearly independent.

Theorem 5.5:

$$f = \sum_{n\epsilon\Gamma}\langle f, \phi_n\rangle\, \tilde{\phi}_n = \sum_{n\epsilon\Gamma}\langle f, \tilde{\phi}_n\rangle\, \phi_n$$

$$\frac{1}{B}\|f\|^2 \le \sum_{n\epsilon\Gamma}\langle f, \tilde{\phi}_n\rangle^2 \le \frac{1}{A}\|f\|^2$$

$$f = \phi_p$$

$$\phi_p = \sum_{n\epsilon\Gamma}\langle \phi_p, \tilde{\phi}_n\rangle\, \phi_n \xrightarrow[\text{independent}]{\text{Linearly}} \langle \phi_p, \tilde{\phi}_n\rangle = \delta[p-n]$$

$$\frac{1}{B}\|\phi_p\|^2 \le \sum_{n\epsilon\Gamma}\langle \phi_p, \tilde{\phi}_n\rangle^2 \le \frac{1}{A}\|\phi_p\|^2 \xrightarrow[(\|\phi_n\|=1)]{\text{If normalized}} A \le 1 \le B$$

Dual Riesz bases are **biorthogonal** families of vectors.

Dual Frame

Dual-Frame Analysis

Theorem 5.5:

$$f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

$\{\phi_n\}_{n\epsilon\Gamma}$ is a frame of the whole signal space and $\{\tilde{\phi}_n\}_{n\epsilon\Gamma}$ its dual frame.

What if $\{\phi_n\}_{n\epsilon\Gamma}$ is a frame of a subspace **V** of the whole signal space and $\{\tilde{\phi}_n\}_{n\epsilon\Gamma}$ its dual frame?

$$? = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

The best linear approximation of f in **V** orthogonal projection of f in **V**

- **Theorem 5.6:** Let $\{\phi_n\}_{n\epsilon\Gamma}$ be a frame of **V** and $\{\tilde{\phi}_n\}_{n\epsilon\Gamma}$ its dual frame in **V**. The orthogonal projection of $f \epsilon \mathbf{H}$ in **V** is

$$P_{\mathbf{V}} f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

Dual Frame

Dual-Frame Analysis

Theorem 5.6:

$$P_{\mathbf{V}}f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n = \sum_{n\epsilon\Gamma} \langle f, \tilde{\phi}_n \rangle \phi_n$$

$$= \tilde{\Phi}^* \Phi f$$

Dual-frame synthesis operator

$$= \Phi^* \tilde{\Phi} f$$

Dual-frame analysis operator

$$P_{\mathbf{V}}f = \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \tilde{\phi}_n$$

$$= \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle (\Phi^* \Phi_{\mathbf{V}})^{-1} \phi_n$$

$$= (\Phi^* \Phi_{\mathbf{V}})^{-1} \sum_{n\epsilon\Gamma} \langle f, \phi_n \rangle \phi_n$$

$$= (\Phi^* \Phi_{\mathbf{V}})^{-1} \Phi^* \Phi f$$

$$P_{\mathbf{V}}f = \sum_{n\epsilon\Gamma} \tilde{\Phi} f[n] \phi_n = \sum_{n\epsilon\Gamma} (\Phi \Phi^*_{\mathbf{Im}\Phi})^{-1} \Phi f[n] \phi_n$$

$$\uparrow$$

$$\tilde{\Phi} f = (\Phi \Phi^*_{\mathbf{Im}\Phi})^{-1} \Phi f$$

$$\uparrow$$

$$\Phi P_{\mathbf{V}} f = \Phi \Phi^* \tilde{\Phi} f = \Phi f$$

- For sparse representation, the selection of $\{\phi_n\}_{n\epsilon\Gamma}$ depends on the signal f, computing the dual frame $\{\tilde{\phi}_n\}_{n\epsilon\Gamma}$ is inefficient.

Frames

2 **Translation-Invariant Dyadic Wavelet Transform**

Dyadic Wavelet Transform

Translation-invariant wavelet transform

- Translation-invariant wavelet dictionaries are constructed by sampling the scale parameter s while keeping all translation parameters u.

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \xrightarrow[\text{Sampling } s]{s = 2^j} \mathcal{D} = \left\{\psi_{u,2^j}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-u}{2^j}\right)\right\}_{u\epsilon\mathbb{R}, j\epsilon\mathbb{Z}}$$

wavelets — Translation-invariant wavelet dictionaries

$$Wf\left(u, 2^j\right) = \langle f, \psi_{u,2^j}\rangle = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2^j}}\psi^*\left(\frac{t-u}{2^j}\right)dt = f * \overline{\psi}_{2^j}(u)$$

- As in the case of orthogonal and biorthogonal wavelet bases, we construct a *scaling function* ϕ and the corresponding *wavelet* ψ with a Fourier transform:

$$\boxed{\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \qquad \hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)}$$

Low-pass FIR filter — band-pass FIR filter

Dyadic Wavelet Transform

Vanishing moments

The number of vanishing moments of ψ

$\shortparallel$

the number of zeros of $\hat{\psi}(\omega)$ at $\omega = 0$.

Apply $\hat{\phi}(0) = 1$ $\longrightarrow$ $\shortparallel$ $\longleftarrow$ Apply $\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$

the number of zeros of $\hat{g}(\omega)$ at $\omega = 0$.

Dyadic Wavelet Transform

Reconstructing Wavelets

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) \quad \xrightarrow{\text{dual}} \quad \hat{\tilde{\phi}}(\omega) = \frac{1}{\sqrt{2}}\hat{\tilde{h}}\left(\frac{\omega}{2}\right)\hat{\tilde{\phi}}\left(\frac{\omega}{2}\right)$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) \quad \xrightarrow{\text{dual}} \quad \hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}}\hat{\tilde{g}}\left(\frac{\omega}{2}\right)\hat{\tilde{\phi}}\left(\frac{\omega}{2}\right)$$

- Reconstructing wavelets are calculated with a pair of finite impulse response dual filters $\tilde{h}$ and $\tilde{g}$.
- What's the condition to guarantee that $\hat{\tilde{\psi}}$ is a reconstruction wavelet ?

■ **Theorem 5.13:** (*Reconstruction condition*) If the filters satisfy

$$\forall \omega \epsilon [-\pi, \pi], \qquad \hat{\tilde{h}}(\omega)\hat{h}^*(\omega) + \hat{\tilde{g}}(\omega)\hat{g}^*(\omega) = 2,$$

then $\hat{\tilde{\psi}}$ is a reconstruction wavelet

Dyadic Wavelet Transform

Spline Dyadic Wavelets

- A *box spline* of degree m: $\hat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right)$ with $\varepsilon = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$

- choosing $\hat{g}(\omega) = O(\omega)$: $\hat{g}(\omega) = -i\sqrt{2}\sin\frac{\omega}{2}\exp\left(\frac{-i\varepsilon\omega}{2}\right)$

The number of vanishing moments of ψ
= The number of zeros of $\hat{g}(\omega)$ at $\omega = 0$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

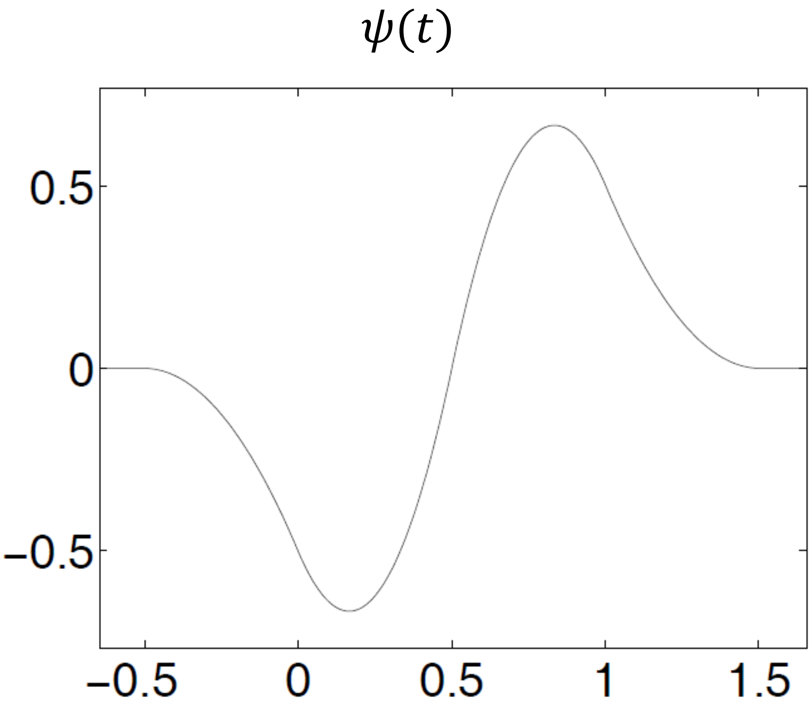
- one vanishing moment wavelet: $\hat{\psi}(\omega) = \frac{-i\omega}{4}\left(\frac{\sin(\omega/4)}{\omega/4}\right)^{m+2} \exp\left(\frac{-i\omega(\varepsilon+1)}{4}\right)$

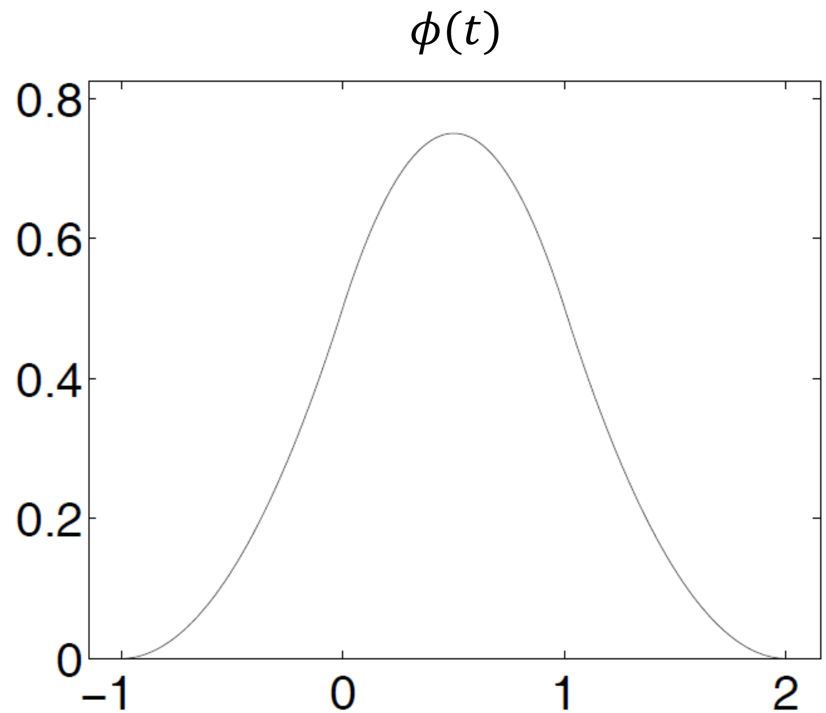
Dyadic Wavelet Transform

Spline Dyadic Wavelets

$$\hat{\psi}(\omega) = \frac{-i\omega}{4}\left(\frac{\sin(\omega/4)}{\omega/4}\right)^{m+2} \exp\left(\frac{-i\omega(\varepsilon+1)}{4}\right), \qquad \hat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right)$$

Set $m = 2, \varepsilon$=1 and take inverse Fourier transform:





Dyadic Wavelet Transform

Fast Dyadic Transform

$a_j[n] = f * \bar{\phi}_{2^j}(n) = \langle f(t), \phi_{2^j}(t-n) \rangle$ with $\phi_{2^j}(t) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{t}{2^j}\right)$

$$a_{j+1}[n] = f * \bar{\phi}_{2^{j+1}}(n)$$

$$\hat{a}_{j+1}(\omega) = \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) \hat{\phi}^*_{2^{j+1}}(\omega + 2k\pi)$$

$$= \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) \sqrt{2^{j+1}} \hat{\phi}^*\left(2^{j+1}(\omega + 2k\pi)\right)$$

Apply $\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} \hat{h}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right)$
and let $\omega = 2^{j+1}(\omega + 2k\pi)$

$$= \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) \sqrt{2^{j+1}} \times \frac{1}{\sqrt{2}} \hat{h}^*\left(2^j (\omega + 2k\pi)\right) \hat{\phi}^*\left(2^j(\omega + 2k\pi)\right)$$

$$= \hat{h}^*\left(2^j \omega\right) \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) \sqrt{2^j} \hat{\phi}^*\left(2^j(\omega + 2k\pi)\right)$$

$$= \hat{h}^*\left(2^j \omega\right) \underbrace{\sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi) \hat{\phi}^*_{2^j}(\omega + 2k\pi)}_{\hat{a}_j(\omega)}$$

$$\boxed{\hat{a}_{j+1}(\omega) = \hat{h}^*\left(2^j \omega\right) \hat{a}_j(\omega)}$$

Similarly, we can also get $\boxed{\hat{d}_{j+1}(\omega) = \hat{g}^*\left(2^j \omega\right) \hat{a}_j(\omega)}$

Dyadic Wavelet Transform

Fast Dyadic Transform

$$\boxed{\hat{a}_{j+1}(\omega) = \hat{h}^*(2^j\omega)\hat{a}_j(\omega), \quad \hat{d}_{j+1}(\omega) = \hat{g}^*(2^j\omega)\hat{a}_j(\omega)}$$

$$\hat{a}_{j+1}(\omega)\hat{\tilde{h}}(2^j\omega) + \hat{d}_{j+1}(\omega)\hat{\tilde{g}}(2^j\omega)$$

$$= \hat{a}_j(\omega)\hat{h}^*(2^j\omega)\hat{\tilde{h}}(2^j\omega) + \hat{a}_j(\omega)\hat{g}^*(2^j\omega)\hat{\tilde{g}}(2^j\omega)$$

$$= \hat{a}_j(\omega)\left[\hat{h}^*(2^j\omega)\hat{\tilde{h}}(2^j\omega) + \hat{g}^*(2^j\omega)\hat{\tilde{g}}(2^j\omega)\right]$$

$$= 2\hat{a}_j(\omega)$$

Theorem 5.13: (*Reconstruction condition*)

$$\forall \omega \epsilon [-\pi, \pi],$$

$$\hat{\tilde{h}}(\omega)\hat{h}^*(\omega) + \hat{\tilde{g}}(\omega)\hat{g}^*(\omega) = 2$$

$$\boxed{\hat{a}_j(\omega) = \frac{1}{2}\left[\hat{a}_{j+1}(\omega)\hat{\tilde{h}}(2^j\omega) + \hat{d}_{j+1}(\omega)\hat{\tilde{g}}(2^j\omega)\right]}$$

Dyadic Wavelet Transform

Fast Dyadic Transform

$$\hat{a}_{j+1}(\omega) = \hat{a}_j(\omega)\hat{h}^*\left(2^j\,\omega\right), \quad \hat{d}_{j+1}(\omega) = \hat{a}_j(\omega)\hat{g}^*\left(2^j\,\omega\right)$$

$$\hat{a}_j(\omega) = \frac{1}{2}\left[\hat{a}_{j+1}(\omega)\hat{\tilde{h}}\left(2^j\,\omega\right) + \hat{d}_{j+1}(\omega)\hat{\tilde{g}}\left(2^j\,\omega\right)\right]$$

Inverse Fourier Transform

- **Theorem 5.14:** for any $j \geq 0$,

$$a_{j+1}[n] = a_j * \bar{h}_j[n], \quad d_{j+1}[n] = a_j * \bar{g}_j[n],$$

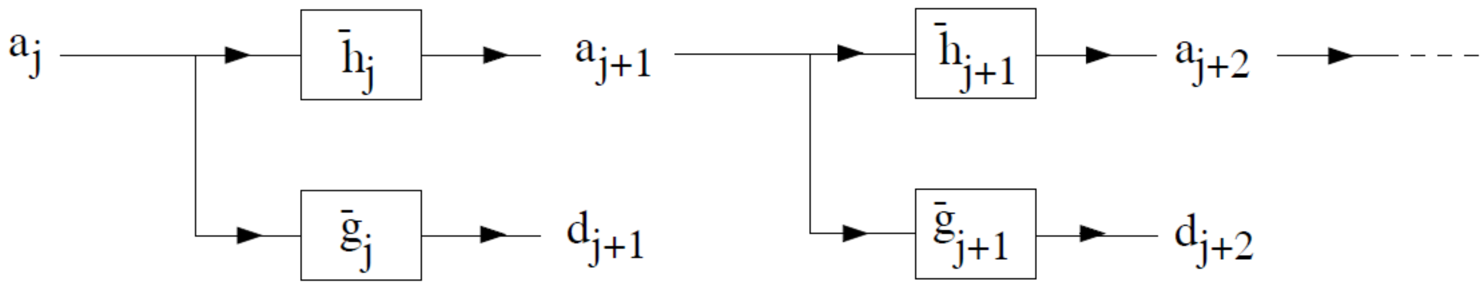
and

$$a_j[n] = \frac{1}{2}\left(a_{j+1}\tilde{h}_j[n] + d_{j+1}\tilde{g}_j[n]\right)$$

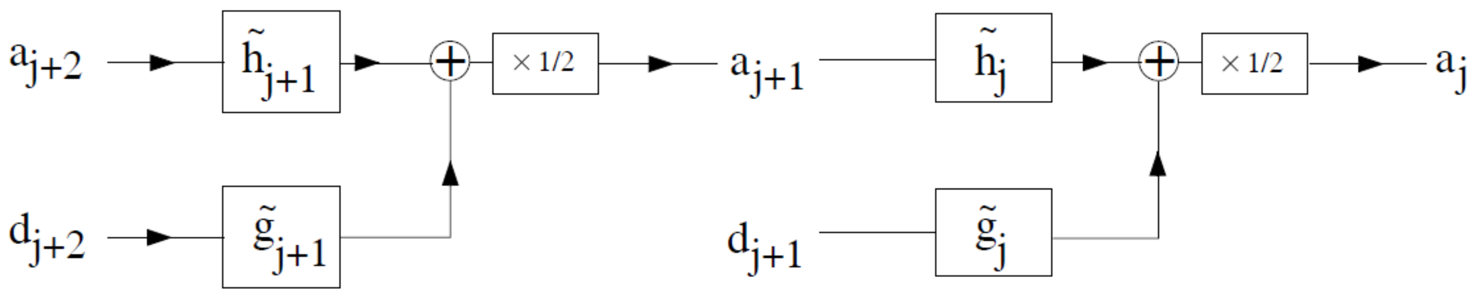
Filter $h_j[n]$ is obtained by inserting $2^j - 1$ zeros between each sample of $h[n]$, its Fourier transform is $\hat{h}(2^j\,\omega)$.

Dyadic Wavelet Transform

Fast Dyadic Transform



$$a_{j+1}[n] = a_j * \bar{h}_j[n], \quad d_{j+1}[n] = a_j * \bar{g}_j[n]$$



$$a_j[n] = \frac{1}{2}\left(a_{j+1}\tilde{h}_j[n] + d_{j+1}\tilde{g}_j[n]\right)$$

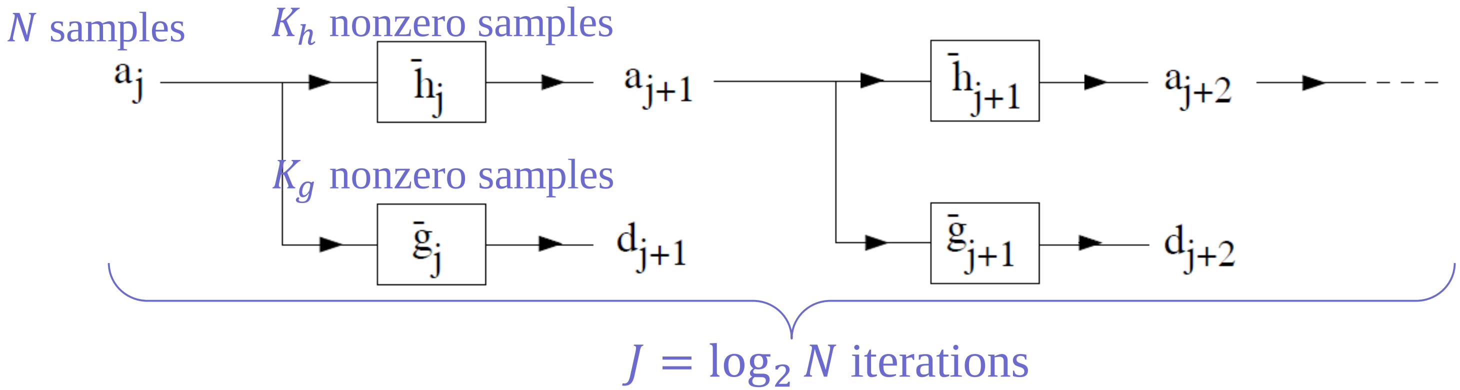
Dual Frame

Fast Dyadic Transform

- The *dyadic wavelet representation* of a_j is defined as the set of wavelet coefficients up to a scale 2^J plus the remaining low-frequency information a_J:

$$[\{d_j\}_{1\le j\le J}, a_J]$$

- **Complexity** :
  - If the input signal $a_0[n]$ has N samples, The maximum scale $2^J = N$, and $J = \log_2 N$.
  - Suppose that h and g have, respectively, K_h and K_g nonzero samples, h_j and g_j have the same nonzero coefficients.
  - The calculation complexity is $(K_h + K_g) N \log_2 N$.



Frames

3

Subsampled Wavelet Frames

Subsampled Wavelet Frames

Fast Dyadic Transform

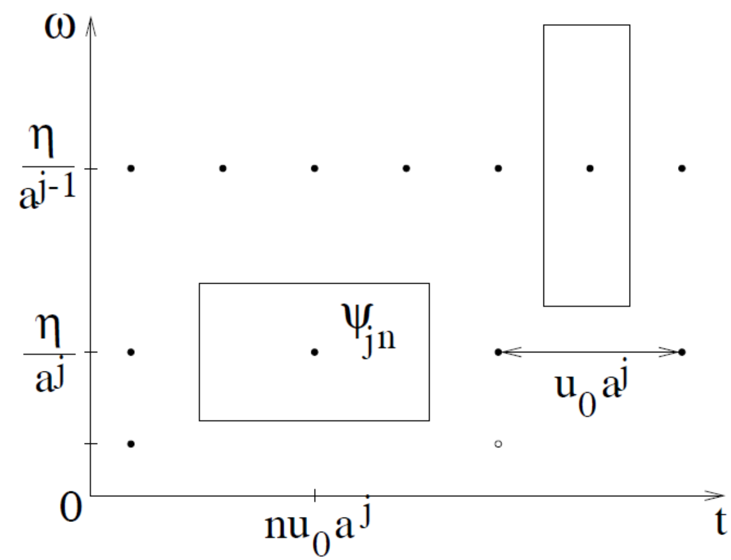
$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \xrightarrow{s = 2^j} \mathcal{D} = \left\{\psi_{u,2^j}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-u}{2^j}\right)\right\}_{u\epsilon\mathbb{R}, j\epsilon\mathbb{Z}}$$

wavelets — Translation-invariant dyadic wavelet dictionaries

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \xrightarrow[u = nu_0 a^j]{s = a^j} \mathcal{D} = \left\{\psi_{j,n}(t) = \frac{1}{\sqrt{a^j}}\psi\left(\frac{t-nu_0a^j}{a^j}\right)\right\}_{(j,n)\epsilon\mathbb{Z}^2}$$

wavelets — Wavelet Frames

- Intuitively, to construct a frame we need to cover the time-frequency plane with the Heisenberg boxes of corresponding discrete wavelet family.
- What's the conditions on ψ, a and u_0 so that $\{\psi_{j,n}(t)\}_{(j,n)\epsilon\mathbb{Z}^2}$ is a frame of $\mathbf{L}^2(\mathbb{R})$?



Subsampled Wavelet Frames

Necessary Conditions

- We suppose that ψ is real, normalized, and satisfies the admissibility condition:

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} \mathrm{d}\omega < +\infty$$

- **Theorem 5.15:** Necessary Condition. If $\{\psi_{j,n}(t)\}_{(j,n)\epsilon\mathbb{Z}^2}$ is a frame of $\mathbf{L}^2(\mathbb{R})$, then the frame bounds satisfy

$$A \leq \frac{C_\psi}{u_0 \ln a} \leq B,$$

$$\forall \omega \epsilon \mathbb{R} - \{0\}, \qquad A \leq \frac{1}{u_0} \sum_{j=-\infty}^{\infty} |\hat{\psi}(a^j \omega)|^2 \leq B$$

- There are continuously differentiable wavelets that generate frames

- In the general case, the dual frame of a wavelet frame is not a wavelet frame

Windowed Fourier Frames

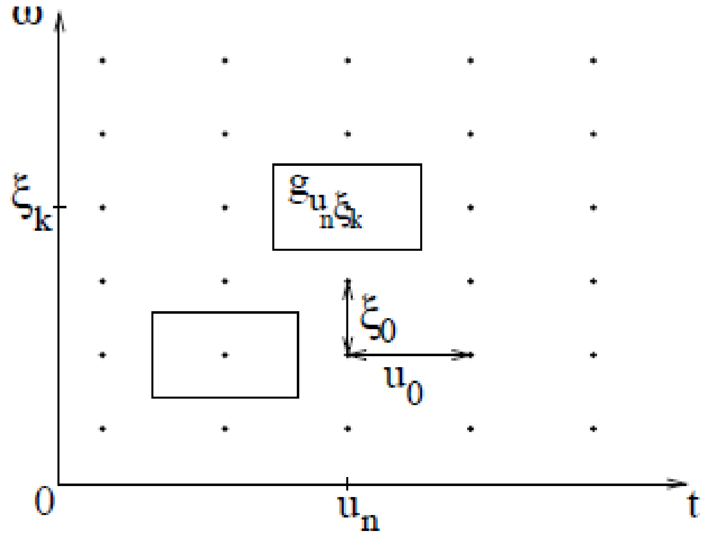
Discretization

$$g_{u,\xi}(t) = g(t-u)\mathrm{e}^{j\xi t} \xrightarrow[\xi = k\xi_0]{u = nu_0} \mathcal{D} = \left\{g_{n,k}(t) = g(t - nu_0)\mathrm{e}^{ik\xi_0 t}\right\}_{(n,k)\epsilon\mathbb{Z}^2}$$

Windowed Fourier Atom

Windowed Fourier Frames

- The size of the Heisenberg box of g_{u_n,ξ_k} is independent of u_n and ξ_k. It depends on the time-frequency spread of the window g. A complete cover of the plane is obtained by translating these boxes over a uniform rectangular grid.



Windowed Fourier Frames

Necessary Condition

- **Theorem 5.19:** Necessary Condition. The windowed Fourier family $\{g_{n,k}\}_{(n,k)\epsilon\mathbb{Z}^2}$ is a frame of $\mathbf{L}^2(\mathbb{R})$ only if

$$\frac{2\pi}{u_0\xi_0} \geq 1$$

The frame bounds A and B necessarily satisfy

$$A \leq \frac{2\pi}{u_0\xi_0} \leq B,$$

$$\forall t\epsilon\mathbb{R}, \qquad A \leq \frac{2\pi}{\xi_0}\sum_{n=-\infty}^{+\infty}|g(t-nu_0)|^2 \leq B$$

$$\forall \omega\epsilon\mathbb{R}, \qquad A \leq \frac{1}{u_0}\sum_{k=-\infty}^{+\infty}|\hat{g}(\omega-k\xi_0)|^2 \leq B$$

- There is no compactly supported, continuously differentiable window that generates an orthogonal windowed Fourier basis (Balian-Low theorem)

- The dual frame of a windowed Fourier frame is also a window Fourier frame

Wavelet Frames And Windowed Fourier Frames

Translation invariance

- In both cases, the frame representation has the drawback of not being translation invariant with respect to time or frequency. Most interesting signal patterns are not naturally synchronized with frame intervals. In particular, the structure of a signal may be degraded at the lower resolutions

- This motivates the study of the dyadic wavelet transform, which is discrete in scale but not in time

- In practice, the dyadic wavelet transform is implemented by perfect reconstruction filter banks. These fast filter banks correspond to wavelet bases which are built from **multiresolution approximations**

Homework

problem 5.13 and 5.15(a)(b) (A Wavelet Tour of Signal Processing, 3rd edition)

Institute of Media,
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Q & A

Many Thanks